TITLE: Proportional Frequency Designs

PROPORTIONAL FREQUENCY DESIGNS

Sidney Addelman
Research Triangle Institute

CONDITION OF EQUAL FREQUENCIES. In 1945 Finney [5] introduced the procedure, known as fractional replication, which permitted the uncorrelated estimation of some of the effects and interactions when only a fraction of the full factorial arrangement was used. The standard method of constructing fractional replicate plans is to first choose an identity relationship and then deduce from this relationship the appropriate treatment combinations. By utilizing the assumption that the higher order interaction effects are negligible this standard procedure permits the estimation of the remaining effects. For the symmetrical factorial structure (all factors having the same number of levels) the standard procedure yields uncorrelated estimates due to the condition of equal frequencies of the factor levels. If the treatment combinations of the 2^5 factorial plan were inspected one would find that

(1) Each level of every factor occurs exactly eight times with every level of any other factor.

(2) Each combination of levels of any factor occurs exactly four times with every combination of levels of each pair of factors.

(3) Each combination of levels of any pair of factors occurs exactly two times with every combination of levels of any other pair of factors.

(4) Each level of any factor occurs exactly two times with every combination of levels of any three factors.

(5) Each level of any factor occurs exactly once with each combination of levels of any four factors.

(6) Each combination of levels of any pair of factors occurs exactly once with every combination of levels of any three factors.

Because the condition of equal frequencies is satisfied for all six of the above cases uncorrelated estimates of all effects can be obtained.

Now consider a 1/4 replicate of the 2^5 factorial structure defined by the identity relationship

$$I = ADE = BCD = ABCE$$

and consisting of the following treatment combinations:

A	B	C	D	E
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	0	1

One can verify that in this plan each level of any factor occurs exactly twice with every level of any other factor, and hence uncorrelated estimates of all main effects are obtainable, if all interactions are negligible. It can also be verified that each level of a factor does not occur the same number of times with every combination of levels of those pairs of factors with which it is aliased. Hence not all main effect estimates are uncorrelated with two-factor interaction estimates.

When one wishes to construct fractional replicate plans for symmetrical factorial arrangements one need only satisfy the appropriate equal frequency conditions to obtain uncorrelated estimates of the effects. However, in the construction of fractional replicate plans for asymmetrical factorial arrangements (all factors not having the same number of levels) the condition of equal frequencies requires more treatment combinations than are necessary to yield uncorrelated estimates.

CONDITION OF PROPORTIONAL FREQUENCIES. Although the equal frequency condition is sufficient to guarantee orthogonality of factors it is not a necessary condition. It was proved by Addelman and Kempthorne [3] that a necessary and sufficient condition that the main effect estimates of two factors be uncorrelated is that the levels of one factor occur with each of the levels of the other factor with proportional frequencies. Consider two factors, A and B, occurring at r and s levels respectively. Let

N = number of treatment combinations in the plan

$n_{i.}$ = number of times the i level of factor A occurs

$n_{.j}$ = number of times the j level of factor B occurs

n_{ij} = number of treatment combinations in which the i level of factor A occurs with the j level of factor B.

The above necessary and sufficient condition for orthogonality can be displayed mathematically as

(1) $$n_{ij} = n_{i.} n_{.j}/N \quad .$$

The condition of proportional frequencies can be generalized so that plans may be constructed with permit uncorrelated estimates of two-factor interactions as well as main effects. Consider three factors A, B and C. In order that the interaction AB can be uncorrelated with C, each combination of the levels of A and B must occur with the levels of C with proportional frequencies, that is

(2) $$n_{ijk} = n_{ij.} n_{..k}/N \, .$$

Since it is desirable that A be uncorrelated with B

(3) $$n_{ij.} = n_{i..} n_{.j.}/N$$

and hence

(4) $$n_{ijk} = n_{i..} n_{.j.} n_{..k}/N^2 \quad .$$

This condition which assures that AB is uncorrelated with C also implies that AC is uncorrelated with B, BC uncorrelated with A, and hence AB, AC and BC are pairwise uncorrelated. If a plan contains four or more factors condition (4) must be replaced by

(5) $$n_{ijkm} = n_{i...} n_{.j..} n_{..k.} n_{...m}/N^3$$

which is the necessary and sufficient condition that a plan permit uncorrelated estimation of all main effects and two-factor interaction effects.

COLLAPSING OF LEVELS. A factor at s_1 levels may be collapsed to a factor at $s_2 < s_1$ levels by making a many-one correspondence of the set of s_1 levels to the set of s_2 levels. If $s_1 = s_2^m$ then the s_1 levels can be collapsed to $(s_1-1)/(s_2-1)$ factors each having s_2 levels. Some illustrations of typical correspondence schemes are presented below.

Three-level factor		Two-level factor
0	$\longrightarrow$	0
1	$\longrightarrow$	1
2	$\longrightarrow$	0

Four-level factor		Three-level factor		Two-level factors
0	$\longrightarrow$	0	$\longrightarrow$	000
1	$\longrightarrow$	1	$\longrightarrow$	011
2	$\longrightarrow$	2	$\longrightarrow$	101
3	$\longrightarrow$	1	$\longrightarrow$	110

Five-level factor		Four-level factor		Three-level factor		Two-level factor
0	$\longrightarrow$	0	$\longrightarrow$	0	$\longrightarrow$	0
1	$\longrightarrow$	1	$\longrightarrow$	1	$\longrightarrow$	1
2	$\longrightarrow$	2	$\longrightarrow$	2	$\longrightarrow$	1
3	$\longrightarrow$	3	$\longrightarrow$	2	$\longrightarrow$	1
4	$\longrightarrow$	0	$\longrightarrow$	0	$\longrightarrow$	0

An orthogonal main-effect plan for the $2^2 \times 3^2$ experiment which permits uncorrelated estimates of all main effects with only nine treatment combinations is now constructed to illustrate the technique of collapsing levels. First construct an orthogonal main-effect plan for four factors, each having three levels with nine treatment combinations, namely

0	0	0	0
0	1	1	2
0	2	2	1
1	0	1	1
1	1	2	0
1	2	0	2
2	0	2	2
2	1	0	1
2	2	1	0

If each of the first two factors are collapsed to two-level factors, the resulting treatment combinations constitute an orthogonal main-effect plan for the $2^2 \times 3^2$ experiment and are displayed below.

0	0	0	0
0	1	1	2
0	0	2	1
1	0	1	1
1	1	2	0
1	0	0	2
0	0	2	2
0	1	0	1
0	0	1	0

The smallest plan which yields uncorrelated estimates of the main effects of the $2^2 \times 3^2$ experiment and which also satisfies the equal frequency condition would require 36 treatment combinations.

It should be mentioned that the proportional frequency condition will be satisfied no matter what type of correspondence scheme is used to perform the collapsing procedure. However, the efficiency of the estimates depends upon the particular correspondence scheme chosen.

If the $(s_i - 1)$ degrees of freedom for each of the t_i factors at s_i levels are represented by $(s_i - 1)$ orthogonal contrasts among the s_i levels, the estimates obtained by these contrasts will be uncorrelated with the estimates obtained with the contrasts for any other factor, because the correspondence scheme automatically guarantees proportional frequencies of the levels of each factor.

REPLACING FACTORS. The collapsing procedure given above can be reversed so that a factor at s^m levels can replace $(s^m - 1)/(s - 1)$ factors, each at s levels. The replacement procedure can be illustrated by the construction of an orthogonal main-effect plan for the 3×2^4 experiment with eight trials. First construct an orthogonal main-effect plan for the 2^7 experiment with eight trials. The seven two-level factors can be represented by X_1, X_2, X_1X_2, X_3, X_1X_3, X_2X_3 and $X_1X_2X_3$.

The treatment combinations for this plan are

0	0	0	0	0	0	0
0	0	0	1	1	1	1
0	1	1	0	0	1	1
0	1	1	1	1	0	0
1	0	1	0	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	0	1	0	0	1

It is known that there exists an orthogonal main-effect plan for the 2^3 experiment with four trials. The treatment combinations for this plan are 000, 011, 101, and 110. Thus, by choosing three factors of the 2^7 plan whose X representations are such that the generalized interaction of any two of the three factors is the third factor, three two-level factors can be replaced by a four-level factor, according to the following correspondence scheme:

Two-level factor				Four-level factor
0	0	0	$\longrightarrow$	0
0	1	1	$\longrightarrow$	1
1	0	1	$\longrightarrow$	2
1	1	0	$\longrightarrow$	3

Since the X representations of the first three factors of the above plan are X_1, X_2 and X_1X_2, these three factors can be replaced by a four-level factor and the orthogonal main-effect plan for the 4×2^4 experiment in eight trials is given by the following treatment combinations:

0	0	0	0	0
0	1	1	1	1
1	0	0	1	1
1	1	1	0	0
2	0	1	0	1
2	1	0	1	0
3	0	1	1	0
3	1	0	0	1

The plan for the 3×2^4 experiment is then obtained by collapsing the four-level factor to a three-level factor by the correspondence

Four-level factor		Three-level factor
0	$\longrightarrow$	0
1	$\longrightarrow$	1
2	$\longrightarrow$	2
3	$\longrightarrow$	1

The smallest plan which yields uncorrelated estimates of main effects in the 3×2^4 experiment and which also satisfies the equal frequency condition would require 24 treatment combinations.

The procedure for constructing plans which permit uncorrelated estimates of all main effects and some or all of the two-factor interaction effects for asymmetrical factorial arrangements consists of first constructing the corresponding plan for a symmetrical factorial arrangement and then utilizing the collapsing or replacing techniques to obtain the desired plan. Whereas a plan permitting uncorrelated estimates of all main effects and all two-factor interactions among the two-level factors in the $2^3 \times 3^4$ experiment would require 72 treatment combinations to satisfy the condition of equal frequencies it would only require 27 treatment combinations to satisfy the proportional frequency condition.

BLOCKING. Even though the proportional frequency designs are highly fractionated they may still require more trials than can be carried out under uniform conditions. Thus, it would be desirable to divide the experimental data into smaller blocks in such a manner that the main effects may still be estimated without correlation. In order to perform an experiment in blocks one may utilize one or more of the factors of an orthogonal main-effect plan for the $4 \times 3^2 \times 2^6$ experiment with sixteen trials. The following plans may be derived from this one by using various factors as blocking factors:

(i) $4 \times 3^2 \times 2^5$ in 2 blocks of 8 treatment combiantions,

(ii) $4 \times 3^2 \times 2^3$ in 4 blocks of 4 treatment combinations,

(iii) $3^2 \times 2^6$ in 4 blocks of 4 treatment combinations,

(iv) $4 \times 3 \times 2^6$ in 4 blocks of 4 treatment combinations.

ORTHOGONAL POLYNOMIALS. The orthogonal contrasts which define effects and interactions in an equal frequency design can be readily determined from a table of orthogonal polynomials. The advantage of using orthogonal contrasts to define effects and interactions arises from the fact that orthogonal polynomials are so constructed that any term of the polynomial is independent of any other term. This property of independence permits one to compute each regression coefficient independently of the others and also facilitates testing the significance of each coefficient:

Tables of orthogonal polynomials for the case of equally spaced levels are readily available, e.g. Fisher and Yates [6], Anderson and Houseman [4]. It would be an impossible task to compile a general table of orthogonal polynomials for unequally spaced levels. However a simple procedure for computing these orthogonal polynomials is available and will be presented below. If equally spaced levels do not each occur in a plan an equal number of times the published tables of orthogonal polynomials are not appropriate. The orthogonal polynomials for equally spaced levels which do not occur in a plan with equal frequency can be computed by the following method for unequally spaced levels.

For any set of orthogonal polynomials the linear contrast is of the form $\Sigma(\alpha + \beta x)y_x$, where α and β are constants, x is the level at which the factor occurs, y_x is the response to the treatment combination with the factor at the x level and the summation is over every value of x which is presented. The quadratic and cubic contrasts are of the form $\Sigma(\alpha + \beta x + \gamma x^2)y_x$ and $\Sigma(\alpha + \beta x + \gamma x^2 + \delta x^3)y_x$, respectively. The extension to higher order contrasts is obvious. Two contrasts are orthogonal if the coefficients of each contrast sum to zero and the sum of products of the corresponding coefficients of the two contrasts is zero.

Table 1

Coefficients of Orthogonal Contrasts

Level of x	Linear	Quadratic	Cubic
0	α	α	α
1	$\alpha + \beta$	$\alpha + \beta + \gamma$	$+\beta + \gamma + \delta$
2	$\alpha + 2\beta$	$\alpha + 2\beta + 4\gamma$	$+2\beta + 4\gamma + 8\delta$
4	$\alpha + 4\beta$	$\alpha + 4\beta + 16\gamma$	$+4\beta + 16\gamma + 64\delta$

We will illustrate the procedure for obtaining orthogonal polynomials for unequally spaced levels with an example.

Consider an independent variable x with levels 0, 1, 2 and 4. The coefficients of the linear, quadratic and cubic contrasts for this example are displayed in Table 1. The coefficients of the linear contrast must sum to zero. Thus,

$$4\alpha + 7\beta = 0.$$

Setting $\beta = 1$ we find that $\alpha = -7/4$. In order that the coefficients of the orthogonal contrasts be integers reduced to lowest terms we multiply these coefficients by 4 to obtain $\beta = 4$ and $\alpha = -7$. Substituting $\alpha = -7$ and $\beta = 4$ in the linear contrast given in Table 1, gives the linear coefficients.

Level of x	Coefficient of linear contrast
0	-7
1	-3
2	1
4	9

The coefficients of the quadratic contrast must sum to zero. Hence,

$$4\alpha + 7\beta + 21\gamma = 0$$

The sum of products of the corresponding coefficients of the linear and quadratic contrasts must also equal zero. Thus,

$$35\beta + 145\gamma = 0$$

Solving these two equations to obtain integral values for α, β and γ we obtain $\alpha = 14$, $\beta = -29$ and $\gamma = 7$.

If we substitute these values in the quadratic contrast and reduce the resulting coefficients to lowest terms the coefficients of the quadratic contrast is given by

Level of x	Coefficient of Quadratic contrast
0	7
1	-4
2	-8
4	5

Similarly the sum of the coefficients of the cubic contrast and the sum of products of the corresponding coefficients of the linear and cubic contrasts must each equal zero. Hence,

$$\begin{aligned} 4\alpha + 7\beta + 21\gamma + 73\delta &= 0 \\ 35\beta + 145\gamma + 581\delta &= 0 \\ 44\gamma + 252\delta &= 0 \end{aligned}$$

Solving these equations to obtain integral values for α, β, γ and we obtain $\alpha = -36$, $\beta = 392$, $\gamma = -315$ and $\delta = 55$. If we substitute these values in the form of the coefficients of the cubic contrast given in Table 1 and reduce the resulting coefficients to lowest terms, the coefficients of the cubic contrast are given by

Level of x	Coefficients of Cubic contrast
0	-3
1	8
2	-6
4	1

The orthogonal polynomials are presented in the following table.

Table 2
Orthogonal Polynomials

Level of x	Linear	Quadratic	Cubic
0	-7	7	-3
1	-3	-4	8
2	1	-8	-6
4	9	5	1

The symbol β represents one unit of the linear effect of a factor when set equal to unity. In order to obtain integral coefficients β was set equal to 4 and hence $(1/4)\beta$ represents one unit of the linear effect. Consequently the linear contrast with coefficients given in Table 2 represents the estimate of 1/4 the linear effect of the factor. It is easily verified that the coefficients of the quadratic contrast are given by

$$7 - \frac{29}{2}\ x + \frac{7}{2}x^2$$

where $x = 0$, 1, 2 and 4 respectively. Thus the symbol $\frac{2}{7}\gamma$ represents one unit of the quadratic effect, and the linear contrast with coefficients given in Table 2 represents the estimate of $\frac{2}{7}$ the quadratic effect of the factor.

Similarly it may be demonstrated that the cubic contrast with coefficients given in Table 2 represents the estimate of 12/55 the cubic effect of the factor.

This constant which is multiplying each effect will be denoted by $1/\lambda$ and in the tables of orthogonal polynomials presented by Addelman and Kempthorne [3] the value of λ and the sum of squares of the coefficients were both given. Any contrast defined by the coefficients given in the tables of orthogonal polynomials represents $1/\lambda$ times the appropriate effect of the factor.

EFFICIENCIES. Although any many-one correspondence of the set of s_1 levels to the set of s_i levels will yield proportional frequencies of the levels, there arises the problem of which correspondence is "best" in some sense. The problem may be solved by determining the efficiencies of the main-effect estimates obtained using proportional frequencies relative to the estimates which would result from using equal frequencies of the levels of each factor.

As an illustration we will calculate the relative efficiency of a three-level factor in a main-effect plan with twenty-five trials.

Assume the correspondence scheme used to collapse a five-level factor to three levels is as follows:

Five-level factor		Three-level factor
0	$\longrightarrow$	0
1	$\longrightarrow$	1
2	$\longrightarrow$	2
3	$\longrightarrow$	2
4	$\longrightarrow$	0

The levels 0, 1, and 2 occur in the ratio's 2: 1: 2. Thus for this factor the 0 level occurs in ten treatment combinations, the 1 level occurs in five treatment combiantions and the 2 level occurs in ten treatment combinations.

The variance of the linear effect estimate of this factor is equal to

$\sigma^2/20$ and hence the information on a unit basis is equal to $20/25\sigma^2 = 4/5\sigma^2$. The variance of the linear effect estimate of a three-level factor in 3^n trials is equal to $\sigma^2/2.3^{n-1}$ and the information on a unit basis is $2.3/^{n-1}/3^n\sigma^2 = 2/3\sigma^2$. Hence the relative efficiency of the linear effect estimate is equal to $(4/5)x(3/2) = 6/5$.

The variance of the quadratic effect estimate for the three-level factor in twenty-five trials is equal to $\sigma^2/4$ and the information is then equal to $4/25\sigma^2$. The variance of the quadratic effect estimate with 3^n trials is equal to $\sigma^2/2.3^{n-2}$ and hence the information on a unit basis is equal to $2/9\sigma^2$. The relative efficiency of the quadratic effect estimate is therefore equal to $(4/25)x(9/2) = 18/25$.

Table 3

Relative Efficiencies of Proportional Frequency Estimates

Level	0		1	Efficiency
	Proportional		frequency	
	1	:	2	8/9
	2	:	3	24/25
	1	:	4	16/25
	3	:	4	48/49
	2	:	5	40/49
	1	:	6	24/49

Level	0 1 2	
Contrast	Proportional frequency	
Linear	1 : 2 : 1	3/4
Quadratic	1 : 2 : 1	9/8
Linear	2 : 1 : 2	6/5
Quadratic	2 : 1 : 2	18/25
Linear	1 : 3 : 1	3/5
Quadratic	1 : 3 : 1	27/25
Linear	2 : 3 : 2	6/7
Quadratic	2 : 3 : 2	54/49
Linear	3 : 1 : 3	9/7
Quadratic	3 : 1 : 3	27/49
Linear	1 : 5 : 1	3/7
Quadratic	1 : 5 : 1	45/49

The relative efficiencies of the estimated effects are presented for various proportional frequencies in Table 3. One would choose the proportional frequencies which give the greatest efficiency of estimates. Thus for example, if an experiment in twenty-five trials involved two-level factors the two levels should occur in the ratio 2 : 3 rather than in the ratio 1 : 4 because the efficiency of the 2 : 3 ratio is 24/25 whereas the efficiency of the 1 : 4 ratio is only 16/25.

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